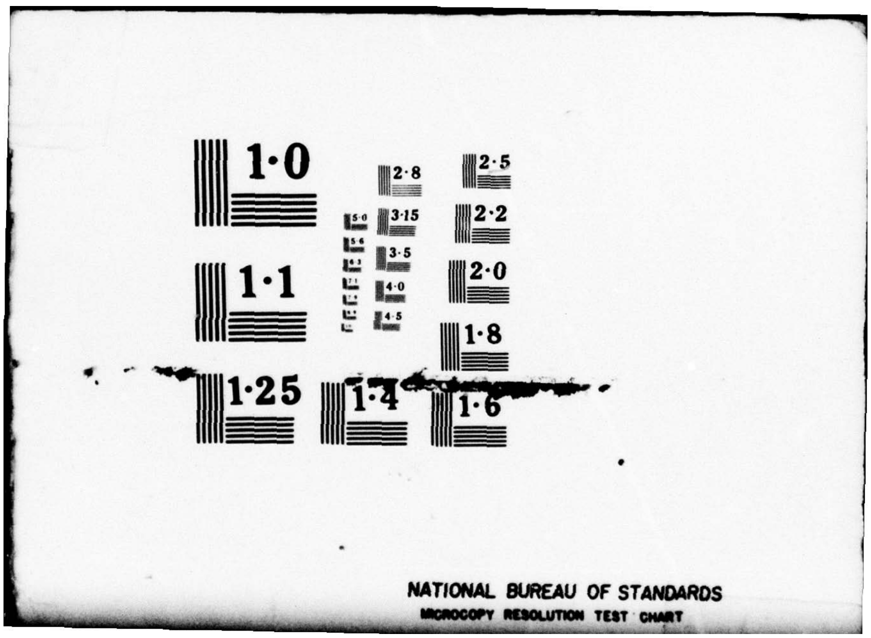
AD-A054 628 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OHIO F/G 20/11
TRAVELING LOADS ON THE TIMOSHENKO BEAM.(U)
APR 78 P J TORVIK
UNCLASSIFIED AFIT-TR-78-2 NL

1 OF 1
ADA
054628

END
DATE
FILMED
6-78
DDC


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·2
2·5
2·8
3·15
3·5
4·0
4·5
5·0
NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

FOR FURTHER ~~TRAN~~

AD A054628

AD No.

DDC FILE COPY

aeH

1

DDC

RECEIVED

JUN 5 1978

B

UNITED STATES AIR FORCE

AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

CIVIL ENGINEERING SCHOOL

DISTRIBUTION STATEMENT A

Approved for public release;
Distribution Unlimited

1

TRAVELING LOADS ON THE TIMOSHENKO BEAM

by

Peter J. Torvik
Professor of Mechanics

AFIT TR 78-2 April 1978

TRAVELING LOADS ON THE TIMOSHENKO BEAM

by

Peter J. Torvik
Professor of Mechanics

AFIT TR 78-2
April 1978

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 14 AFIT-TR-78-2

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle) 6 TRAVELING LOADS ON THE TIMOSHENKO BEAM.

5. TYPE OF REPORT & PERIOD COVERED 9 Technical Rep[redacted]rt.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) 10 Peter J. Torvik

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Air Force Institute of Technology
Wright-Patterson AFB, Ohio 45433

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 12 42p.

11. CONTROLLING OFFICE NAME AND ADDRESS
Air Force Institute of Technology
Wright-Patterson AFB, Ohio 45433

12. REPORT DATE 11 Apr[redacted]78

13. NUMBER OF PAGES 41

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report) UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Approved for public release; IAW AFR 190-17

JERREL F. GUESS, Captain, USAF
Director of Information

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Beam Vibrations
Moving Load
Timoshenko Beam

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A transverse force traveling along an infinite string or a beam at critical values of constant velocity generates unbounded amplitudes, in the absence of dissipation. This resonance is analagous to the unbounded amplitudes generated by a stationary force oscillating at one of the natural frequencies. The response of a finite elementary beam to a moving force of constant amplitude can be determined in terms of the eigenfunctions of the beam. Modification of elementary beam theory to take into account the effects of rotatory inertia and → next page
CONTINUED

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

012 200

shear leads to the Timoshenko beam theory, from which a new set of eigenvalues and eigenfunctions can be determined. These eigenfunctions can be shown to have an orthogonality relationship which, although unusual, permits the solution of initial value and non-homogeneous problems. The procedure for solving such problems is given, and applied to the problem of a traveling load on a finite Timoshenko beam with arbitrary end conditions. Results are obtained for the case of pinned ends, and compared with those from elementary theory. Results of particular significance are that the distribution of critical speeds is altered significantly through inclusion of rotatory and shear effects, and that a shear wave, not present in the results from elementary theory, can be identified and is shown to play a major role in determining the response.

Preface

Stephan Prokofievitch Timoshenko was born on December 22, 1878 in the Ukraine. After making substantial contributions to applied mechanics in Russia, he left Russia in 1920 for Serbia, and in 1922 emigrated to this country. After five years in industry, (Vibration Specialty Company and Westinghouse), he joined the faculty of the University of Michigan as a professor of graduate mechanics and later (1936) accepted an appointment at Stanford University.

The subject of this study, Traveling Loads on the Timoshenko Beam, seems particularly appropriate in this, the centenary of Timosheko's birth. In addition to developing the improved beam theory, (*Philosophical Magazine*, 1921-1922) which now bears his name, Timoshenko's childhood interest in railway engineering was fulfilled through many works on the response of rails and bridges to moving and pulsating loads. Equation 110 of this paper, the response of a Bernoulli-Euler beam to a moving force was given by Timoshenko in the Bulletin of the Polytechnical Institute of Kiev in 1908 and is included in his *Vibration Problems in Engineering*.

In view of Professor Timoshenko's interest in this subject it is somewhat surprising that the comparisons made here were not given long ago by Timoshenko himself.

Table of Contents

List of Figures

Abstract

A transverse force travelling along an infinite string or a beam at critical values of constant velocity generates unbounded amplitudes, in the absence of dissipation. This resonance is analagous to the unbounded amplitudes generated by a stationary force oscillating at one of the natural frequencies. The response of a finite elementary beam to a moving force of constant amplitude can be determined in terms of the eigenfunctions of the beam. Modification of elementary beam theory to take into account the effects of rotatory inertia and shear leads to the Timoshenko beam theory, from which a new set of eigenvalues and eigenfunctions can be determined. These eigenfunctions can be shown to have an orthogonality relationship which, although unusual, permits the solution of initial value and non-homogeneous problems. The procedure for solving such problems is given, and applied to the problem of a travelling load on a finite Timoshenko beam with arbitrary end conditions. Results are obtained for the case of pinned ends, and compared with those from elementary theory. Results of particular significance are that the distribution of critical speeds is altered significantly through inclusion of rotatory and shear effects, and that a shear wave, not present in the results from elementary theory, can be identified and is shown to play a major role in determining the response.

I. Introduction

The Timoshenko beam has been considered in many studies, and has found numerous applications since it was first presented[1]. Comparisons have been made between the predictions of this more exact theory and the predictions of Bernoulli-Euler theory for the natural frequencies of finite beams, and for the transient response of beams of infinite or semi-infinite length[2,3]. While a number of techniques have been applied to the solution of problems involving the Timoshenko beam, eigenfunction expansions do not appear to have received much attention in the solution of initial value problems, and in the determination of the forced response.

It is evident that the eigenfunctions of the Timoshenko beam are orthogonal in the usual manner in the case of simply supported ends. Although a non-classic orthogonality for other boundary conditions has been demonstrated[4,5], and it has been shown that arbitrary initial conditions can be satisfied by a series of the eigenfunctions[5], these results do not appear in the standard texts on structural vibrations. It also does not appear to be well known that the response to a distributed time dependent force can be written in terms of the eigenfunctions. Thus it appears appropriate to set down a formulation of the orthogonality condition which leads directly to a procedure whereby initial-value and prescribed force problems for the Timoshenko beam may be treated. This procedure will then be applied to two example problems, the transient response to a uniform impulse, and to the subject problem.

II. Analysis of Timoshenko Beam

The development of the equations of motion will first be outlined, primarily as a means of defining the terms to be employed. The deflections of the beam are taken to be

$$u_x(x,z,t) = - z \; \psi \; (x,t) \tag{1}$$

$$u_z(x,z,t) = \omega(x,t) \tag{2}$$

plus whatever small displacements arise through Poisson contractions. Such displacements will be presumed to give rise to no strain energy, nor any kinetic energy. The beam geometry, and the sign conventions for displacements and loads are depicted in Figure 1. It is to be noted that a distributed moment, not present in elementary theory has been provided for, although it is not likely to occur in practice.

From the figure, and Equations 1 and 2,

$$\varepsilon_{xx}(x,z,t) = - z\psi_{,x} \tag{3}$$

$$\varepsilon_{xz}(x,z,t) = \frac{1}{2}(\omega_{,x}-\psi) \tag{4}$$

The comma here denotes partial differentiation with respect to the coordinate.

Applying the stress strain law leads to a uniform shear stress which, when integrated over the section, leads to a pseudo shear force

$$V' = \iint_A \sigma_{xz}\, dA = \iint_A \mu(\omega_{,x}-\psi)\, dA = \mu A(\omega_{,x}-\psi) \tag{5}$$

with μ being the shear modulus. Since the actual shear stress is not uniform, it is customary to introduce a correction factor κ, leading to an expression for the actual shear force,

$$V(x,t) = \mu\kappa A(\omega_{,x}-\psi) \tag{6}$$

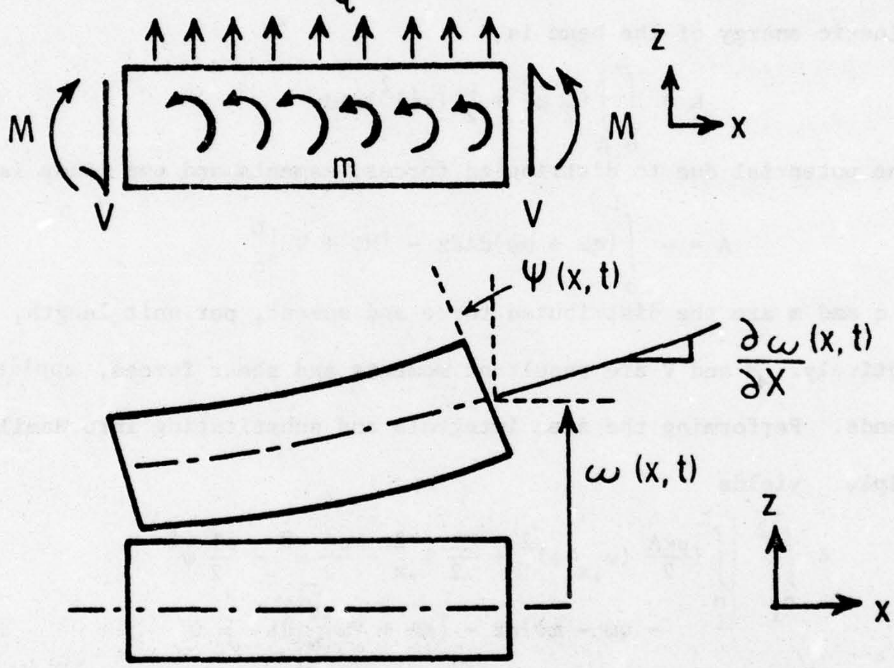


Figure 1 Deflection and Loading of the Timoshenko Beam

Various estimates of the shear correction factor have been given[6]. Taking the energy due to the shear to be the product of the average shear stress, V/A, and the uniform shear strain, the total strain energy in the beam is found to be

$$U = \int_o^L \int_A \{\frac{1}{2} \kappa\mu(\omega_{,x} - \psi)^2 + \frac{E}{2}(z\psi_{,x})^2\} dAdx \tag{7}$$

The length of the beam is taken as L, and the Young's modulus to be E. The kinetic energy of the beam is

$$K = \int_o^L \int_A \{\frac{\rho}{2} \dot{\omega}^2 + \frac{\rho}{2} (z\dot{\psi})^2\} dAdx \tag{8}$$

and the potential due to distributed forces, moments and end loads is

$$A = - \int_o^L \{q\omega + m\psi\} dAdx - |M\psi + V\omega|_o^L \tag{9}$$

where q amd m are the distributed force and moment, per unit length, respectively. M and V are resultant moments and shear forces, applied at the beam ends. Performing the area integrals and substituting into Hamilton's principle yields

$$\delta \int_{t_1}^{t_2} \left[\int_o^L \{\frac{\mu\kappa A}{2} (\omega_{,x} - \psi)^2 + \frac{EI}{2} \psi_{,x}^2 - \frac{\rho A}{2} \dot{\omega}^2 - \frac{\rho I}{2} \dot{\psi}^2 - q\omega - m\psi\} dx - |M\psi + V\omega|_o^L \right] dt = 0 \tag{10}$$

where $I = \int z^2 dA$. Taking the variation, integrating by parts, and collecting terms leaves two Euler equations and two boundary conditions.

$$- EI\psi_{,xx} - \mu\kappa A(\omega_{,x} - \psi) + \rho I\ddot{\psi} - m = 0 \tag{11}$$

$$- \mu\kappa A(\omega_{,x} - \psi)_{,x} + \rho A\ddot{\omega} - q = 0 \tag{12}$$

$$|[\mu\kappa A(\omega_{,x} - \psi) - V]\delta\omega|_o^L = 0 \tag{13}$$

$$|(EI\psi_{,x} + \psi)\delta\psi|_o^L = 0 \tag{14}$$

Modulus, area, and moment of inertia have been taken as uniform, although the extension to nonuniform sections is not difficult. Thus, satisfying

a prescribed shear force or displacement and moment or rotation at each end, together with satisfaction of the differential equations on $0 \leq x \leq L$ leads to an extremum of the time integral of the Lagrangian function.

III. Solution

It is convenient to put Equations 11-12 in a dimensionless form. To this end, we introduce a dimensionless length,

$$\eta = x/r \quad \text{where} \quad r^2 = I/A \tag{15}$$

a dimensionless time

$$\tau = tc/r \quad \text{where} \quad c^2 = \mu\kappa/\rho \tag{16}$$

and dimensionless displacements, moment per unit length, and force per unit length through the definitions

$$\phi = \omega/r \tag{17}$$

$$M = m/\mu\kappa A \tag{18}$$

$$Q = qr/\mu\kappa A \tag{19}$$

The elastic properties can then be described by a single quantity

$$\gamma = \frac{E}{\kappa\mu} = \frac{2(1+\nu)}{\kappa} \tag{20}$$

The result of these substitutions into Equations 11 and 12 are dimensionless forms of the governing differential equations,

$$\gamma\psi_{,\eta\eta} + \phi_{,\eta} - \psi - \psi_{,\tau\tau} + M = 0 \tag{21}$$

$$\phi_{,\eta\eta} - \psi_{,\eta} - \phi_{,\tau\tau} + Q = 0 \tag{22}$$

The homogeneous boundary conditions become:

$$(\phi_{,\eta} - \psi) \text{ or } \phi \text{ must vanish, and} \tag{23}$$

$$\psi_{,\eta} \text{ or } \psi \text{ must vanish} \tag{24}$$

at $\eta = 0$ and $\eta = L/r = \ell$. It is convenient to rewrite Equations 21 and 22 as a single matrix equation,

$$AU_{,\eta\eta} + BU_{,\eta} + CU - DU_{,\tau\tau} + F = 0 \tag{25}$$

where

$$U = \begin{Bmatrix} \psi \\ \phi \end{Bmatrix} . \tag{26}$$

$$F = \begin{Bmatrix} M \\ Q \end{Bmatrix} \quad \text{and} \tag{27}$$

$$A \equiv \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}, \; B \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; C \equiv \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \; D \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{28}$$

We first address the homogeneous problem, $F \equiv 0$. Let

$$U = \begin{Bmatrix} \Psi(\eta)\theta(\tau) \\ \Phi(\eta)\theta(\tau) \end{Bmatrix} = U(\eta)\theta(\tau) \tag{29}$$

Here U is a vector of the two independent displacements, but U^TDU is a scalar and we may obtain a separation of variables by division.

$$\frac{U^TAU_{,\eta\eta} + U^TBU_{,\eta} + U^TCU}{U^TDU} = \frac{1}{\theta}\frac{d^2\theta}{d\tau^2} = -\lambda \tag{30}$$

The differential equation to be satisfied by the eigenfunctions is

$$AU_{,\eta\eta} + BU_{,\eta} + CU + \lambda DU = 0 \tag{31}$$

All homogeneous boundary conditions satisfying equation 23 and 24 may be put into the form

$$B_1U_{,\eta} + B_2U = 0 \tag{32}$$

and must be satisfied at 0 and ℓ. The necessary matrices B_i are:

For a clamped end:

$$B_1 \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; B_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

For a simply supported end:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{34}$$

For a free end:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \tag{35}$$

and for a guided end:

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \tag{36}$$

Satisfaction of the matrix differential equation (31) with a boundary condition for each end selected from 33-36 permits the eigenfunctions to be determined.

A. Wave Propagation

The propagation of a sinusoidal wave train in a beam of infinite length has been the subject of a number of studies[6]. The motion is then assumed to be

$$U = \begin{Bmatrix} A \\ B \end{Bmatrix} \exp j\,(kx - \Omega t) \tag{37}$$

In terms of the dimensionless variables used here, we have

$$\begin{Bmatrix} \psi \\ \phi \end{Bmatrix} = \begin{Bmatrix} A \\ B \end{Bmatrix} \exp j\,(kr\eta - \frac{\Omega r}{c}\tau) \tag{38}$$

Satisfaction of equations 21 and 22, with M and Q set to zero requires that

$$\gamma A\,(-k^2r^2) + Bjkr - A + A\,\frac{\Omega^2 r^2}{c^2} = 0 \tag{39}$$

$$B(-k^2r^2) - Ajkr + B\Omega^2 r^2/c^2 = 0 \tag{40}$$

From these, the dispersion relationship or frequency spectrum is obtained;

$$\{\frac{\Omega^2 r^2}{c^2} - 1 - \gamma k^2 r^2\}\{\frac{\Omega^2 r^2}{c^2} - k^2r^2\} - k^2r^2 = 0 \tag{41}$$

and, if desired, the ratio A/B.

In terms of a dimensionless frequency, $\sqrt{\lambda} = \Omega r/c$, and a dimensionless wavenumber, $K = rk$, this becomes

$$\lambda^2 - [1 + (1+\gamma)K^2]\lambda + \gamma K^4 = 0 \tag{42}$$

For any K (wavenumber, or reciprocal wavelength), two values of frequency are found. This dispersion relationship is shown (for $\gamma = 4$) in Figure 2 as the solid lines. For the lower branch, and small values of K, it can be shown that

$$\lambda \cong \gamma K^4$$

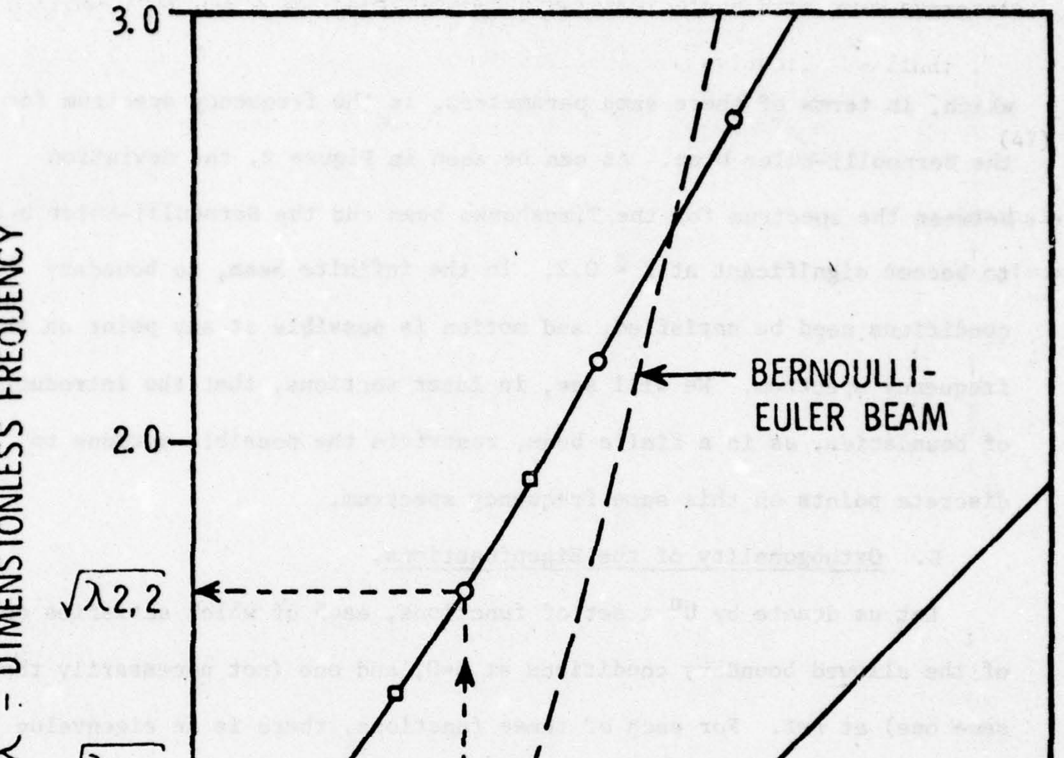


Figure 2 Frequency Spectrum for a Timoshenko Beam, γ = 4.0

which, in terms of there same parameters, is the frequency spectrum for the Bernoulli-Euler beam. As can be seen in Figure 2, the deviation between the spectrum for the Timoshenko beam and the Bernoulli-Euler beam begins to become significant at $K \cong 0.2$. In the infinite beam, no boundary conditions need be satisfied, and motion is possible at any point on the frequency spectrum. We will see, in later sections, that the introduction of boundaries, as in a finite beam, restricts the possible motions to discrete points on this same frequency spectrum.

B. Orthogonality of the Eigenfunctions.

Let us denote by U^n a set of functions, each of which satisfies one of the allowed boundary conditions at $\eta=0$, and one (not necessarily the same one) at $\eta=\ell$. For each of these functions, there is an eigenvalue λ_n, which we assume to be distinct.

Thus, for any n, we have

$$AU^n_{,\eta\eta} + BU^n_{,\eta} + CU^n + \lambda_n DU^n = 0 \tag{44}$$

For some other eigenvalue, $\lambda_m \neq \lambda_n$, we have

$$AU^m_{,\eta\eta} + BU^m_{,\eta} + CU^m + \lambda_m DU^m = 0 \tag{45}$$

Multiplying the first by U^{mT} and the second by U^{nT}, where T denotes the transpose of the 2-vectors, integrating over the length, and subtracting, we have

$$\int_0^\ell \{U^{mT}AU^n_{,\eta\eta} - U^{nT}AU^m_{,\eta\eta}\}d\eta + \int_0^\ell \{U^{mT}BU^n_{,\eta} - U^{nT}BU^m_{,\eta}\}d\eta$$
$$+ \int_0^\ell \{U^{mT}CU^n - U^{nT}CU^m\}d\eta + \int_0^\ell \{\lambda_n U^{mT}DU^n - \lambda_m U^{nT}DU^m\}d\eta = 0 \tag{46}$$

where $\ell = L/r$. Integrating the first by parts leaves only boundary terms, since A is symmetric. Integrating only the second term of the second

integral by parts leaves only boundary terms, since B is skew symmetric. Since C and D are symmetric, the reductions are evident. We find:

$$\left| U^{mT} A U^{n}_{,\eta} - U^{nT} A U^{m}_{,\eta} - U^{nT} B U^{m} \right|_{o}^{\ell} = (\lambda_m - \lambda_n) \int_{o}^{\ell} U^{mT} D U^{n} d_{\eta} \tag{47}$$

Evaluation of the boundary term is done more easily (and more convincingly) by returning (through Equations 28 and 29) to the components of the matrix U. We find the boundary terms to be

$$\left| \left\{ \psi^m, \phi^m \right\} \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \psi^n_{,\eta} \\ \phi^n_{,\eta} \end{Bmatrix} - \left\{ \psi^n, \phi^n \right\} \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \psi^m_{,\eta} \\ \phi^m_{,\eta} \end{Bmatrix} - \left\{ \psi^n, \phi^n \right\} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \psi^m \\ \phi^m \end{Bmatrix} \right|_{o}^{\ell}$$

$$= \left| \psi^m \gamma \psi^n_{,\eta} + \phi^m \phi^n_{,\eta} - \psi^n \gamma \psi^m_{,\eta} - \phi^n \phi^m_{,\eta} + \phi^n \psi^m - \psi^n \phi^m \right|_{o}^{\ell}$$

$$= \left| \psi^m \gamma \psi^n_{,\eta} - \psi^n \gamma \psi^m_{,\eta} + \phi^m (\phi^n_{,\eta} - \psi^n) - \phi^n (\phi^m_{,\eta} - \psi^m) \right|_{o}^{\ell} \tag{48}$$

But, each product vanishes if any of the four boundary conditions given by Equations 13 and 14, or by Equations 33-36, are satisfied at each end. Thus, for $\lambda_m \neq \lambda_n$, we have the desired orthogonality condition in terms of the vector U

$$\int_{o}^{\ell} U^{mT} D U^{n} d\eta = 0 \ , \tag{49}$$

or, in terms of the components,

$$\int_{o}^{\ell} \{\psi^m \psi^n + \phi^m \phi^n\} d\eta = 0 \ , \text{ when } n \neq m. \tag{50}$$

In terms of the physical rotation and displacement, the orthogonality condition may be rewritten

$$\int_{o}^{\ell} \{\Psi^m I \Psi^n + W^m A W^n\} dx = 0 \tag{51}$$

where

$$W^n = r \phi^n \tag{52}$$

An equivalent version of this result has been given[4,5], but does not seem to be widely known.

Although only homogeneous boundary conditions have been discussed, it is easily shown that orthogonality exists (in the same sense) if lineal or torsional springs are applied at the ends, but does not if translatory or rotatory masses are attached to the end(s).

Thus, orthogonality between the eigenfunctions of the Timoshenko beam exists as it does in the Bernoulli-Euler beam, if the eigenfunctions are **viewed as a 2-vectors of displacement and rotation, rather than** as displacements alone.

C. Initial Value Problems.

We now assume that a Timoshenko beam, having appropriate boundary conditions at each end, is released at time zero into free vibration, with specified initial conditions

$$\begin{aligned} \psi(x,o) &= f(x) \\ \omega(x,o) &= g(x) \\ \dot{\psi}(x,o) &= h(x) \\ \dot{\omega}(x,o) &= k(x) \end{aligned} \tag{53}$$

We look for a separable solution, in the form of Equation 29, employing eigenfunctions, given by 44, for each eigenvalue, λ_n. Solutions to the differential equation and boundary conditions exist only at discrete values of the wavenumber, K of the frequency spectrum for the Timoshenko Beam, as noted in Figure 2. For each wavenumber, however, two natural frequencies are found, and may be denoted $\sqrt{\lambda_{n,1}}$ and $\sqrt{\lambda_{n,2}}$, respectively. This is demonstrated by the dashed lines of Figure 2. Thus, for each wavenumber k_m, m = 1,2 ...., ∞, we will find two eigenvalues λ, which we may number

$$\lambda_n = \lambda_{2m-1} = \lambda_{m,1}$$

$$\lambda_n = \lambda_{2m} = \lambda_{m,2}$$

This sequence, while convenient, does not order the eigenvalues in ascending magnitude, as is evident from Figure 2. While two eigenvalues arise from the same wavenumber, they do not generate the same eigenfunction, for the ratio $\Psi^{n,1}/\Phi^{n,1} \neq \Psi^{n,2}/\Phi^{n,2}$. Consequently, $U^{n,1} \neq U^{n,2}$, as will be more evident from examples to be given.

We let

$$\mathcal{U} = \sum_{n=1}^{\infty} \left\{ \begin{matrix} \Psi^n(\eta) \\ \Phi^n(\eta) \end{matrix} \right\} \left(a_n \cos\sqrt{\lambda_n}\,\tau + b_n \sin\sqrt{\lambda_n}\,\tau\right) \tag{55}$$

where $U^n = \left\{ \begin{matrix} \Psi^n \\ \Phi^n \end{matrix} \right\}$ satisfies Equation 44. The satisfaction of initial conditions (Equation 53) requires

$$\psi(\eta,o) = f(\eta) = \sum_{n=1}^{\infty} a_n \Psi^n(\eta) \tag{56}$$

$$\omega(\eta,o) = g(\eta) = r\sum_{n=1}^{\infty} a_n \Phi^n(\eta) \tag{57}$$

$$\dot{\psi}(\eta,o) = h(\eta) = \sum_{n=1}^{\infty} \frac{c}{r} b_n \sqrt{\lambda_n}\, \Psi^n(\eta) \tag{58}$$

$$\dot{\omega}(\eta,o) = k(\eta) = r\sum_{n=1}^{\infty} \frac{c}{r} b_n \sqrt{\lambda_n}\, \Phi^n(\eta) \tag{59}$$

Multiplying Equations 56 and 58 by Ψ^m, Equations 57 and 59 by Φ^m, adding 56 to 57 and 58 to 59, integrating over the beam length, and applying the orthogonality condition, Equation 50, we find

$$a_m = \frac{\int_o^{\ell} \{f(\eta)\Psi^m(\eta) + \frac{1}{r} g(\eta)\Phi^m(\eta)\}d\eta}{\int_o^{\ell} \{(\Psi^n)^2 + (\Phi^n)^2\}d\eta} \tag{60}$$

$$b_m = \frac{\int_o^{\ell} \{rh(\eta)\Psi^m(\eta) + k(\eta)\Phi^m(\eta)\}d\eta}{c\sqrt{\lambda_n}\int_o^{\ell} \{(\Psi^n)^2 + (\Phi^n)^2\}d\eta} \tag{61}$$

Thus, the necessary coefficients for the expansion given by Equation 55 for the initial value problem specified by Equations 53 have been determined. An equivalent result has been previously given[5].

D. Forced Vibrations.

We now return to the general problem, Equation 25, of a beam subjected to an arbitrary, time dependent, load and moment distribution. Each end of the beam is assumed to satisfy an appropriate homogeneous boundary condition. Only a particular solution is required, as the method of the previous section may be used to furnish the homogeneous solution.

We first obtain the eigenfunctions, and expand the load distribution and the dynamic response in these functions, as

$$F = \sum_{n=1}^{\infty} D\, F_n(\tau) U^n(\eta) \tag{62}$$

and

$$\mathcal{U} = \sum_{n=1}^{\infty} G_n(\tau) U^n(\eta) \tag{63}$$

Substituting into Equation 25, multiplying by U^{mT} and integrating over the length, we find

$$\begin{aligned}\int_0^{\ell} \{\sum_n U^{mT} A G_n(\tau) U^n_{,\eta\eta} + \sum_n U^{mT} B G_n(\tau) U^n_{,\eta} + \sum_n U^{mT} C G_n(\tau) U^n\} d\eta \\ - \int_0^{\ell} \sum_n U^{mT} D G_n''(\tau) U^n d\eta + \int_0^{\ell} \sum_n U^{mT} F_n(\tau)\, D\, U^n d\eta = 0\end{aligned} \tag{64}$$

The first three terms may be evaluated by recognizing that they contain the first three terms of Equation 44. We find

$$\sum_n \int_0^{\ell} \{-\lambda_n U^{mT} D U^n G_n(\tau)\} - U^{mT} D U^n G_n''(\tau) + U^{mT} D U^n F_n(\tau)\} = 0 \tag{65}$$

Since the U^n are orthogonal, these equations uncouple, and we have

$$G_n''(\tau) + \lambda_n G_n(\tau) = F_n(\tau) \tag{66}$$

where $F_n(\tau)$, determined from Equation 62, is found to be

$$F_n(\tau) = \int_0^\ell U^{nT} F d\eta \Bigg/ \int_0^\ell U^{nT} D U^n d\eta \tag{67}$$

or

$$F_n(\tau) = \int_0^\ell \{M\Psi^n + Q\Phi^n\} d\eta \Bigg/ \int_0^\ell U^{nT} D U^n d\eta \tag{68}$$

The modal forces, $F_n(\tau)$, may be computed from the original, physical, moments and forces,

$$F_n(\tau) = \frac{\displaystyle\int_0^\ell \{\frac{m}{\mu\kappa A}(\eta,\tau)\Psi^n(\eta) + \frac{rq}{\mu\kappa A}(\eta,\tau)\Phi^n(\eta)\} d\eta}{\displaystyle\int_0^\ell \{(\Psi^n)^2 + (\Phi^n)^2\} d\eta} \tag{69}$$

Solving the set of ordinary equations (66) for the scalar time dependent coefficients $G_n(\tau)$ and substituting into (63) completes the solution for the forced response. The method of Laplace transforms and convolution is particularly easily to apply. If the initial values of ω, $\dot{\omega}$, ψ and $\dot{\psi}$ are all zero, then the initial values of $G_n(\tau)$ and $\dot{G}_n(\tau)$ must all be zero. Taking transforms of Equation (66) then yields

$$s^2 g_n(s) + \lambda_n g_n(s) = f_n(s) \tag{70}$$

Hence,

$$G_n(\tau) = \int_0^\tau \frac{\sin}{\sqrt{\lambda_n}} (\sqrt{\lambda_n}(\tau - p)) F_n(p) dp \tag{71}$$

and

$$\begin{Bmatrix} \psi(\eta,\tau) \\ \omega(\eta,\tau) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} \Psi^n(\eta) \\ r\Phi^n(\eta) \end{Bmatrix} \left\{ \int_0^\tau \frac{\sin}{\sqrt{\lambda_n}} (\sqrt{\lambda_n}(\tau-\theta)) \frac{\displaystyle\int_0^\ell \{\frac{m(\eta,\theta)}{\mu\kappa A}\Psi^n(\eta) + \frac{rq(\eta,\theta)}{\mu\kappa A}\Phi^n(\eta)\} d\theta}{\displaystyle\int_0^\ell \{(\Psi^n)^2 + (\Phi^n)^2\} d\eta} \right\} \tag{72}$$

and the forced response is determined.

If the initial values of displacements and velocities are not zero, the free response may be determined by the method of the previous section and superposed, or the initial conditions may be expanded in the series given as Equation (63), $G_n(0)$ and $G_n'(0)$ determined, and included in the Transformation, Equation (70). A similar result has been given[5].

IV. Examples

A. Initial Value Problem.

We consider a uniform, simply supported, Timoshenko beam, subjected to an impulse I, per unit length of the beam at time t=0. Hence, at t=0^+, the beam has an initial velocity, given by

$$V_o = I/\rho A \tag{73}$$

Thus

$$k(x) = I/\rho A, \; f(x) = g(x) = h(x) = 0 \tag{74}$$

The appropriate boundary conditions for the simply supported beam are that

$$\phi(o) = \phi(\ell) = \psi_{,\eta}(o) = \psi_{,\eta}(\ell) = o \tag{75}$$

To find the eigenfunctions, we let

$$U = \begin{Bmatrix} A \\ B \end{Bmatrix} e^{p\eta} \tag{76}$$

and substitute into 21 and 22 to arrive at a characteristic equation

$$\lambda^2 + \lambda[(1 + \gamma)p^2 - 1] + \gamma p^4 = o \tag{77}$$

and the ratio of amplitudes:

$$A/B = (\lambda + p^2)/p \tag{78}$$

For any λ, there are four values of p satisfying equation 77, which we denote $p_1 = ir_1, p_2 = -ir_1, p_3 = ir_2, p_4 = -ir_2$, where

$$r_1 = \sqrt{\{\lambda(\gamma + 1) - \sqrt{\lambda^2(\gamma - 1)^2 + 4\gamma\lambda}\,\}/2\gamma}$$

$$r_2 = \sqrt{\{\lambda(\gamma + 1) + \sqrt{\lambda^2(\gamma - 1)^2 + 4\gamma\lambda}\,\}/2\gamma} \tag{79}$$

and four corresponding ratios, $(A/b)_i$. With some redesignation of terms, we find

$$\Phi = b_1 \sin r_1\eta + b_2 \cos r_1\eta + b_3 \sin r_2\eta + b_4 \cos r_2\eta \tag{80}$$

$$\begin{aligned} \Psi = &-b_1 \left(\frac{\lambda - r_1^2}{r_1}\right)\cos r_1\eta + b_2\left(\frac{\lambda - r_1^2}{r_1}\right)\sin r_1\eta \\ &-b_3 \left(\frac{\lambda - r_2^2}{r_2}\right)\cos r_2\eta + b_4\left(\frac{\lambda - r_2^2}{r_2}\right)\sin r_2\eta \end{aligned} \tag{81}$$

The boundary conditions for the simply supported beam are satisfied by $b_2 = b_4 = 0$ and either

$$r_1 \ell = m\pi \ , \quad b_3 = 0 \ , \quad m = 1,2 \cdots\cdots \infty$$

or

$$r_2 \ell = m\pi \ , \quad b_1 = 0 \ , \quad m = 1,2 \cdots\cdots \infty$$

Substitution of either of these into Equation 77 leads to the same equation for λ, and substitution into Equations 80 and 81 leads to the same mode shapes; hence they are the same solution. The quantities r_i may be recognized as having the same definition as the wavenumber, K, introduced previously. Thus we see that motion of the finite, simply supported Timoshenko beam can occur only at discrete wavenumbers,

$$K = r = m\pi/\ell \tag{82}$$

For each m, however, there are two eigenvalues, $\lambda_{m,1}$ and $\lambda_{m,2}$, as depicted in Figure 2. The circles in that figure correspond to the eigenvalues for $m = 1,2,3$ and 4 in a beam for which $\ell = 10$, and the open squares denote the eigenvalues for a beam of $\ell = 100$ corresponding to $m = 1, 5, 15$ and 25. From this figure we may also observe that elementary beam theory gives a good approximation to the first 5 frequencies in the longer beam, but only gives a fair approximation of the first in the shorter.

Thus, the eigenfunctions are

$$U^{2m-1} = \begin{Bmatrix} \psi^{2m-1} \\ \phi^{2m-1} \end{Bmatrix} = \begin{bmatrix} \dfrac{(\frac{m\pi}{\ell})^2 - \lambda_{m,1}}{(\frac{m\pi}{\ell})} \cos \dfrac{m\pi\eta}{\ell} \\ \\ \sin \dfrac{m\pi\eta}{\ell} \end{bmatrix}$$

$$U^{2m} = \begin{Bmatrix} \psi^{2m} \\ \phi^{2m} \end{Bmatrix} = \begin{bmatrix} \dfrac{(\frac{m\pi}{\ell})^2 - \lambda_{m,2}}{m\pi/\ell} \cos m\pi \dfrac{\eta}{\ell} \\ \\ \sin \dfrac{m\pi\eta}{\ell} \end{bmatrix} \tag{83}$$

where

$$\lambda_{m,i} = \frac{[1 + (1+\gamma)(\frac{m\pi}{\ell})^2] \mp \left[[1 + (1+\gamma)(\frac{m\pi}{\ell})^2]^2 - 4\gamma(\frac{m\pi}{\ell})^4\right]^{1/2}}{2} \tag{84}$$

The $\lambda_{m,1}$ are evaluated with the negative sign, and $\lambda_{m,2}$ are evaluated using the positive. From these, the quantity

$$\int_0^\ell U^{n^T} D U^n d\eta \tag{85}$$

may be evaluated, with the result

$$\int_0^\ell U^{n^T} D U^n d\eta = \left\{\left[\frac{(\frac{m\pi}{\ell})^2 - \lambda_{m,i}}{(\frac{m\pi}{\ell})}\right]^2 + 1\right\}\frac{\ell}{2} \tag{86}$$

with i=1 when n=2m-1, and i=2 when n=2m. Substitution into Equations 60 and 61, and the result into 55 gives

$$\begin{Bmatrix}\psi \\ \omega/r\end{Bmatrix} = U = \sum_{m=1}^{\infty}\begin{bmatrix}\left[\dfrac{(\frac{m\pi}{\ell})^2 - \lambda_{m,1}}{\frac{m\pi}{\ell}}\right]\cos\dfrac{m\pi\eta}{\ell} \\ \sin\dfrac{m\pi\eta}{\ell}\end{bmatrix}\frac{\frac{1}{c\sqrt{\lambda_{m,1}}}\frac{I}{\rho A}(\frac{\ell}{m\pi})(1-\cos m\pi)\sin\sqrt{\lambda_{m,1}}\,\tau}{\left\{\left[\dfrac{(\frac{m\pi}{\ell})^2 - \lambda_{m,1}}{m\pi/\ell}\right]^2 + 1\right\}\frac{\ell}{2}}$$

$$+ \sum_{m=1}^{\infty}\begin{bmatrix}\left[\dfrac{(\frac{m\pi}{\ell})^2 - \lambda_{m,2}}{m\pi/\ell}\right]\cos m\pi\eta/\ell \\ \sin m\pi\eta/\ell\end{bmatrix}\frac{\frac{1}{c\sqrt{\lambda_{m,2}}}\frac{I}{\rho A}\frac{\ell}{m\pi}(1-\cos m\pi)\sin\sqrt{\lambda_{m,2}}\,\tau}{\left\{\left[\dfrac{(\frac{m\pi}{\ell})^2 - \lambda_{m,2}}{m\pi/\ell}\right]^2 + 1\right\}\frac{\ell}{2}} \tag{87}$$

The displacement, ω, can be simplified significantly,

$$\omega = \frac{4Ir}{\rho A c\ell}\sum_{m=1,3,\cdots\infty}\sum_{i=1}^{2}\frac{\sin m\pi\frac{\eta}{\ell}}{\sqrt{\lambda_{m,i}}}\frac{(\ell/m\pi)(\sin\sqrt{\lambda_{m,i}}\,\tau)}{\left[\left(\dfrac{(m\pi/\ell)^2 - \lambda_{m,i}}{(m\pi/\ell)}\right)^2 + 1\right]} \tag{88}$$

and the expression for the rotation is

$$\psi = \frac{4I}{\rho A c\ell}\sum_{m=1,3\cdots\infty}\sum_{i=1}^{2}\frac{\cos m\pi\frac{\eta}{\ell}}{\sqrt{\lambda_{m,i}}}\frac{\{(\frac{m\pi}{\ell})^2 - \lambda_{m,i}\}(\sin\sqrt{\lambda_{m,i}}\,\tau)}{(\frac{m\pi}{\ell})^2\left[\{\dfrac{(m\pi/\ell)^2 - \lambda_{m,i}}{(m\pi/\ell)}\}^2 + 1\right]} \tag{89}$$

The transverse displacements given by Equation 88 were evaluated for a beam with γ=4 and ℓ=100. The peak displacement at the beam center was found to be

$$\omega_{max} = 162.6 \left(\frac{4\mathcal{J}r}{\rho Ac}\right) \tag{90}$$

occuring at τ = 701. For a Bernoulli-Euler beam of the same geometry and the same loading, the peak response was found to be within 1% of this value, and occurred at τ = 681. The same computations were performed for a shorter beam (ℓ=10), with the results that a peak displacement of

$$\omega_{max} = 1.756 \left(\frac{4\mathcal{J}r}{\rho Ac}\right) \tag{91}$$

was found for the Timoshenko beam, occurring at τ =10.65, and 8% less for the Bernoulli-Euler beam, occurring at τ = 6.8.

The displacement time response at the center of a beam (ℓ=10) are compared for the two theories in Figure 3. A more dramatic difference between the two theories occurs in the predictions for the bending moment. Evaluating the first derivative of Equation 89 at the beam center (η = 5) leads to the moment time history shown as the solid line in Figure 3, while the second derivative of the series for the displacements, i.e., the moment, as determined from the elementary theory does not converge.

B. Traveling Loads on the Timoshenko Beam.

Let us assume that a simply supported Timoshenko beam is subjected to an external load

$$q(x,t) = P\delta(x-vt) \tag{92}$$

where P is a force, moving at speed v, beginning from x=0 at t=0. We assume that the beam displacement and velocity are zero at t=0. Here P is positive in the +z direction of Figure 1. In terms of dimensionless variables, introduced through Equation 19,

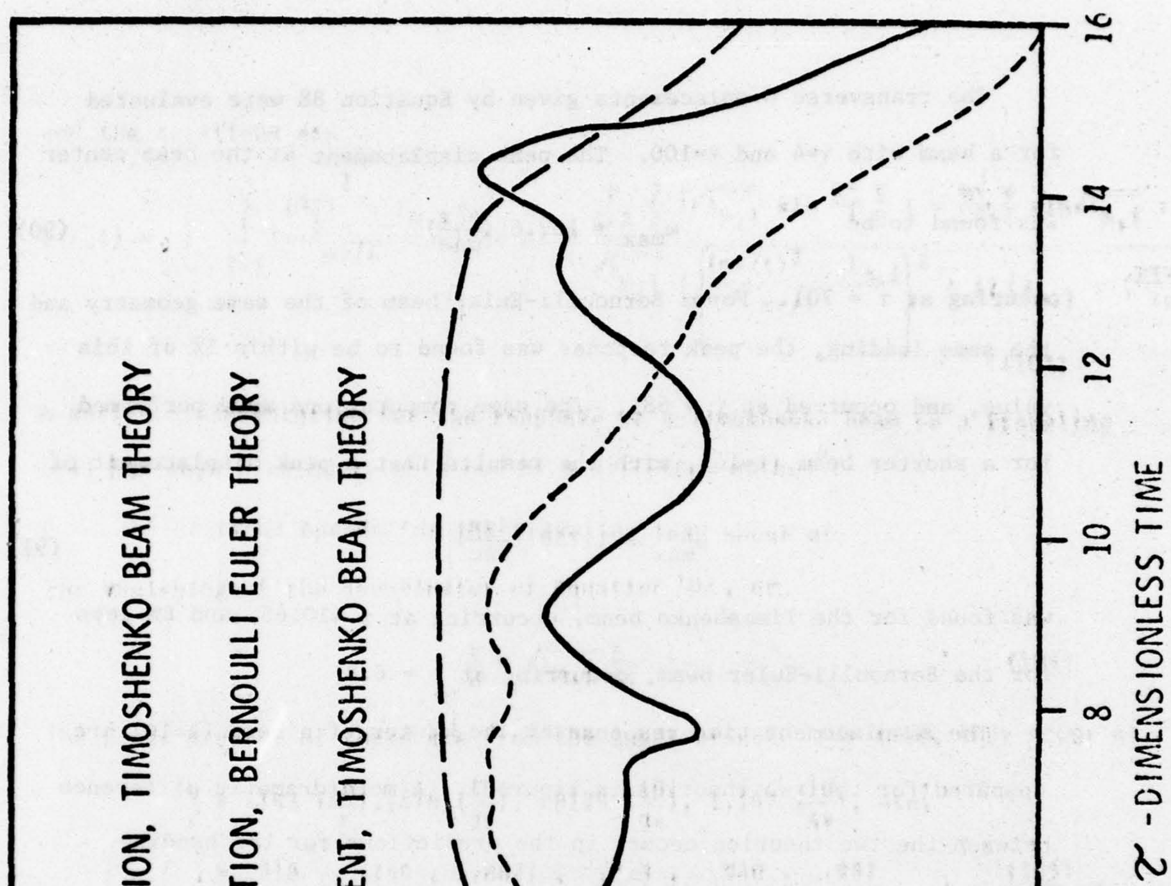


Figure 3 **Response of Uniform Beam to Uniform Impulse, ℓ = 10.**

Deflection and Moment at Beam Center.

$$Q = \frac{qr}{\mu\kappa A} = \frac{r}{\mu\kappa A} P\delta(r\eta - \frac{vr}{c}\tau) \tag{93}$$

and m is taken to be zero. The differential equation to be satisfied is

$$A\mathcal{U}_{,\eta\eta} + B\mathcal{U}_{,\eta} + C\mathcal{U} - D\mathcal{U}_{,\tau\tau} + F = 0 \tag{94}$$

where

$$\mathcal{U} = \begin{Bmatrix} \psi \\ \phi \end{Bmatrix} \quad \text{and} \quad F = \begin{Bmatrix} \frac{rP}{\mu\kappa A}\,\delta(r\eta - vr\tau/c) \\ 0 \end{Bmatrix} \tag{96}$$

We first expand the applied load in a series of the eigenfunctions,

$$F = \sum_{n=1}^{\infty} D\, F_n(\tau) U^n(\eta) \tag{97}$$

and look for the response in terms of the same eigenfunctions,

$$\mathcal{U} = \sum G_n(\tau) U^n(\eta) \tag{98}$$

The response is that given by Equation 72.

The modal forces are given by Equation 69, or

$$F_n(\tau) = \frac{\int_0^{\ell} \{\frac{rP}{\mu\kappa A}\,\delta(r\eta - vr\tau/c)\} \sin n\pi \frac{\eta}{\ell}\, d\eta}{\int_0^{\ell} \{\psi^{n2} + \phi^{n2}\} d\eta} \tag{99}$$

or

$$F_n(\tau) = \{\frac{P}{\mu\kappa A} \sin \frac{n\pi}{\ell}\frac{v\tau}{c}\} / \{\frac{\ell}{2}\left[\left\{\frac{(\frac{m\pi}{\ell})^2 - \lambda_{m,i}}{\frac{m\pi}{\ell}}\right\}^2 + 1\right]\} \tag{100}$$

where the choice i=1 is again used for n=2m-1, and i=2 is used for n=2m. The complete solution is

$$\begin{Bmatrix} \psi(\eta,\tau) \\ \omega(\eta,\tau) \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{i=1}^{2} \begin{Bmatrix} [(\frac{m\pi}{\ell})^2 - \lambda_{m,i}]\frac{\cos m\pi\eta/\ell}{m\pi/\ell} \\ r \sin \frac{m\pi\eta}{\ell} \end{Bmatrix} \frac{P}{\mu\kappa A} \int_0^{\tau} \sin\frac{m\pi v\theta}{\ell c} \frac{\sin[\sqrt{\lambda_{m,i}}(\tau-\theta)]\, d\theta}{\frac{\ell}{2}[(\frac{(\frac{m\pi}{\ell})^2 - \lambda_{m,i}}{m\pi/\ell})^2 + 1]\sqrt{\lambda_{m,i}}} \tag{101}$$

Finally, the solution for displacement may be evaluated as

$$\omega(\eta,\tau) = \sum_{m=1}^{\infty}\sum_{i=1}^{2} \frac{rP}{\mu\kappa A}\frac{2}{\ell} \sin m\pi\frac{\eta}{\ell} \frac{\sqrt{\lambda_{m,i}} \sin \frac{m\pi}{\ell}\frac{v}{c}\tau - \frac{m\pi}{\ell}\frac{v}{c} \sin \sqrt{\lambda_{m,i}}\,\tau]}{[\left\{\frac{(m\pi/\ell)^2 - \lambda_{m,i}}{m\pi/\ell}\right\}^2 + 1]\sqrt{\lambda_{m,i}}\,[\lambda_{m,i} - (\frac{m\pi v}{\ell c})^2]} \tag{102}$$

and the rotations are

$$\psi(\eta,t) = \sum_{m=1}^{\infty} \sum_{i=1}^{2} \left[\frac{(\frac{m\pi}{\ell})^2 - \lambda_{m,i}}{m\pi/\ell}\right] \cos m\pi \frac{\eta}{\ell} \frac{\frac{P}{\mu\kappa A}\frac{2}{\ell}}{\sqrt{\lambda_{m,i}}} \frac{\{\sqrt{\lambda_{m,i}} \sin \frac{m\pi}{\ell}\frac{v}{c}\tau - \frac{m\pi}{\ell}\frac{v}{c} \sin\sqrt{\lambda_{m,i}}\,\tau\}}{\left[\left\{\frac{(m\pi/\ell)^2 - \lambda_{m,i}}{m\pi/\ell}\right\}^2 + 1\right]\left[\lambda_{m,i} - (\frac{m\pi v}{\ell c})^2\right]} \tag{103}$$

A series representation for the response of a Timoshenko beam to a traveling load has been previously obtained by a different method[8].

The critical speeds for the traveling load occur at the vanishing of the denominator of equation 102 , or

$$\frac{v}{c} = \sqrt{\lambda_{m,i}}\,\frac{\ell}{m\pi} \tag{104}$$

The first several of these are from the lower branch. For $\ell=10$, they occur at

$$\frac{v}{c} = .163\ (\frac{10}{\pi}), .476\ (\frac{10}{2\pi}),\ .8125\ (\frac{10}{3\pi}),\ 1.147\ (\frac{10}{4\pi}), \text{ etc.}$$

$$= .518\ ,\ \ .756\ ,\ \ .8621\ ,\ \ .913\ ,\ \ .940\ ,\ \ .957 \tag{105}$$

It can be shown that these values assymptotically approach 1. The critical frequencies from the upper branch occur at

$$\frac{v}{c} = 1.211\ (\frac{10}{\pi}),\ 1.657\ (\frac{10}{2\pi}),\ 2.187\ (\frac{10}{3\pi}),\ 2.753\ (\frac{10}{4\pi}), \text{ etc.}$$

$$= 3.854\ ,\ \ 2.637\ ,\ \ 2.320\ ,\ \ 2.191\ , \text{ etc.} \tag{106}$$

These critical values asymptotically approach 2.0, or in general, $\sqrt{\gamma}$. For a longer beam, say $\ell=100$, the critical speeds from the first branch occur at

$$\frac{v}{c} = .001969\ (\frac{100}{\pi}),\ .00782\ (\frac{100}{2\pi}),\ .01739\ (\frac{100}{3\pi}),\ .03042\ (\frac{100}{4\pi}), \text{ etc.}$$

$$\text{or } \frac{v}{c} = .0627\ ,\ \ .1244\ ,\ \ .1845\ ,\ \ .2421\ , \text{ etc.} \tag{107}$$

and asymptotically approach 1 from below. For the critical speeds resulting from upper beanch frequencies, the first four critical speeds are

$$\frac{v}{c} = 1.002\left(\frac{100}{\pi}\right),\ 1.010\left(\frac{100}{2\pi}\right),\ 1.022\left(\frac{100}{3\pi}\right),\ 1.038\left(\frac{100}{4\pi}\right),\ \text{etc.}$$

$$\text{or}\quad \frac{v}{c} = 31.89,\ 16.07,\ 10.84,\ 8.260,\ \text{etc.} \tag{108}$$

and asymptotically approach 2.0, from above, or in general $\sqrt{\gamma}$.

For short beams, the critical frequencies are densely packed in two narrow bands. For longer beams, they are somewhat more dispersed. In any case, no critical speeds occur in the interval $1 \leq v/c \leq \sqrt{\gamma}$.

The character of the response is strongly governed by the ratio of the load velocity to the critical velocity. For low values of v (Figure 4) the peak displacement occurs ahead of the point of load application at early times. For a load velocity close to the critical value, the peak displacement occurs at the load (Figure 5), while for load velocities much higher than the critical value, the peak displacements are seen to occur well behind the load (Figure 6). In each case, the dot denotes the position of the load at the instant the displacements are given.

The shear velocity c, is also seen to play an important role in the response. In Figures 4, 5 and 6, the location of a front of a wave traveling at speed c is denoted by the short vertical line. In Figures 4 and 5, low and intermediate velocities, the load velocity is below the velocity of the shear wave and no significant displacement occurs in front of the shear wave. In Figure 6, the load velocity is much in excess of the shear velocity, and the peak displacement is seen to occur at the shear front, but is reduced by an order of magnitude.

It is of interest to compare these results to the response of a Bernoulli-Euler beam to the same traveling load, so that the influence of rotatory inertia and shear deformations on the motion may be assessed. The differential equation to be solved is that of the Bernoulli-Euler beam, or

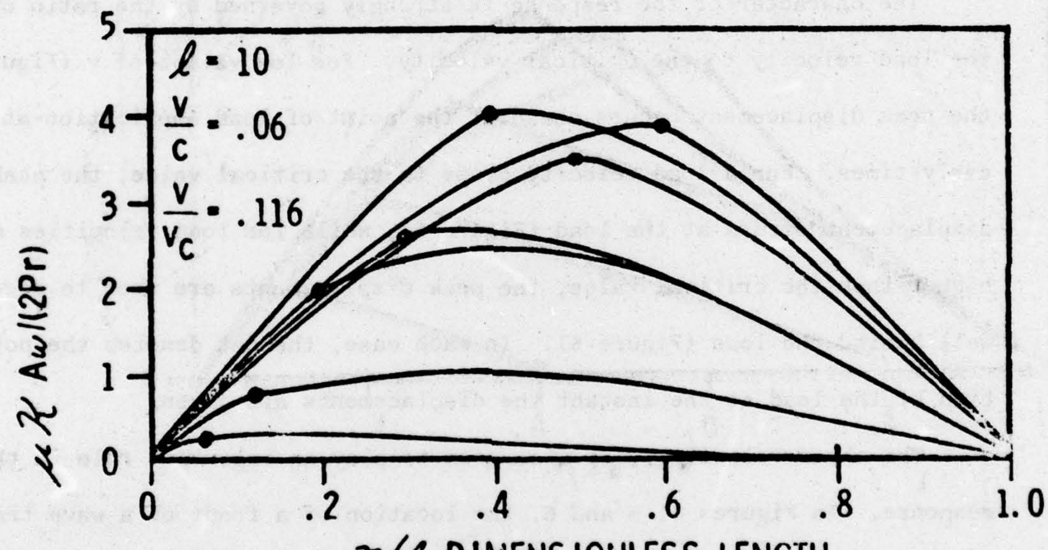


Figure 4 **Displacements of Timoshenko Beam for Low Value of Load Velocity**

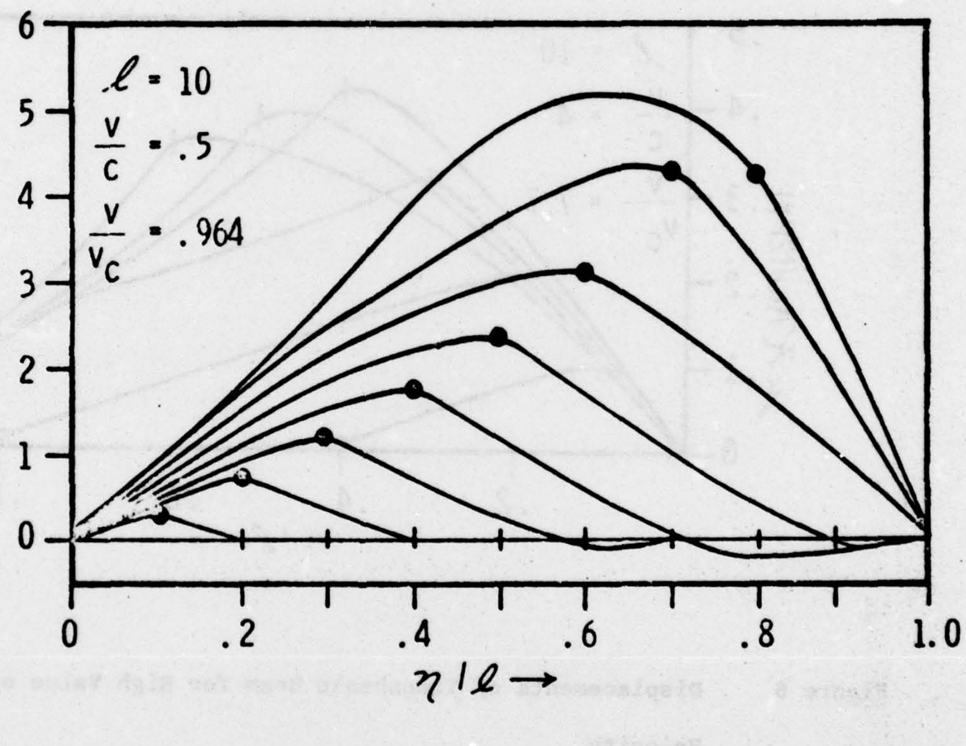


Figure 5 Displacements of Timoshenko Beam for Intermediate Value of Load Velocity

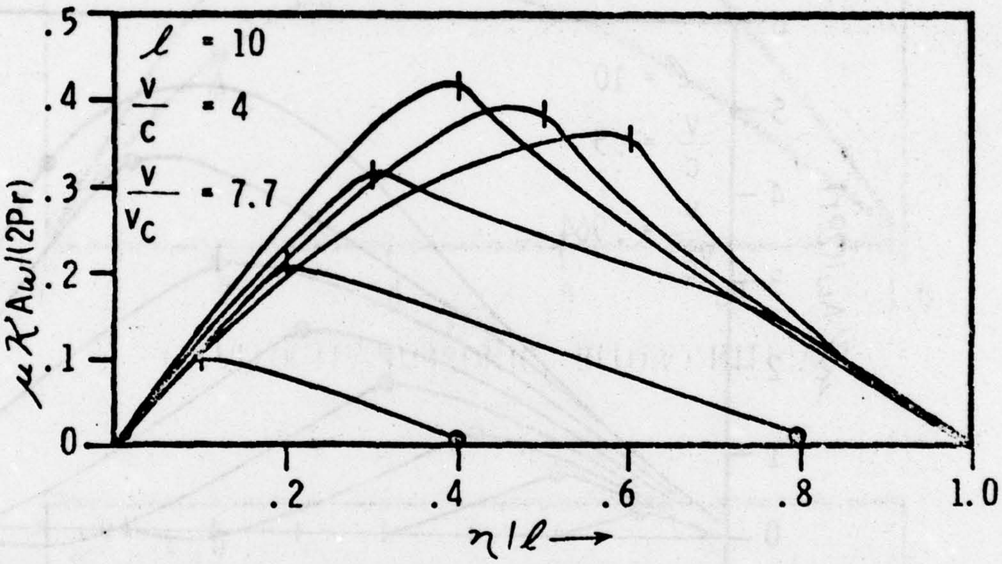


Figure 6 **Displacements of Timoshenko Beam for High Value of Load Velocity**

$$EIw_{xxxx} + \rho A\ddot{\omega} = q = P\delta(x - Vt) \tag{109}$$

The solution, for a simply supported beam of length L, initially at rest, is found to be

$$\omega = \sum_{n=1}^{\infty} \frac{2P}{L\rho A} \frac{\{\sin n\pi vt/L - (n\pi v/L\omega_n)\sin \omega_n t\}\sin n\pi x/L}{\frac{EI}{\rho A} n^4 \frac{\pi^4}{L^4} - n^2\pi^2 v^2/L^2} \tag{110}$$

where $\omega_n{}^2 = EIn^4\pi^4/(\rho AL^4)$ (111)

In order to compare the results of the elementary theory with the results from the Timoshenko beam theory, we make the following substitutions: $\tau = \frac{tc}{r}$, $\eta = x/r$, $\ell = L/r$, $I/A = r^2$, $c^2 = \mu\kappa/\rho$, as before and set

$$\frac{E}{\rho} = \frac{2(1+\nu)}{\kappa} c^2 = \gamma c^2 \tag{112}$$

$$\sqrt{\lambda_n} = \omega_n r/c = \sqrt{\gamma}\, n^2\pi^2/\ell^2 \tag{113}$$

Thus

$$\omega = \frac{2Pr}{\mu\kappa A} \cdot \frac{1}{\ell} \sum_{n=1}^{\infty} \frac{\{\sin \frac{n\pi}{\ell}\frac{v}{c}\tau - \frac{n\pi}{\ell}\frac{v}{c}\frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\tau\}\sin n\pi\eta/\ell}{\lambda_n - \frac{n^2\pi^2}{\ell^2}\frac{v^2}{c^2}} \tag{114}$$

The introduction of the shear coefficient into the equations is totally artificial, done only to facilitate comparison with the Timoshenko beam. The critical speeds arise at the vanishing of the denominator, and are

$$\left.\frac{v}{c}\right)_{crit} = \sqrt{\gamma}\,\frac{n\pi}{\ell} \tag{115}$$

or

$$\left.\frac{v}{c}\right)_{crit} = .628\ ,\ 1.256\ ,\ 1.884\ ,\ 2.512\ ,\ \text{etc.} \tag{116}$$

These velocities, for $\gamma=4$ and $\ell=10$, increase indefinitely, rather than reaching an asymptote, as was found in the case of the Timoshenko beam. For $\ell=100$, the critical values are 10% of those given.

Some computed results are given in Figures 7 and 8. The predictions of the elementary theory, Equation 114, are indistinguishable (Figure 7)

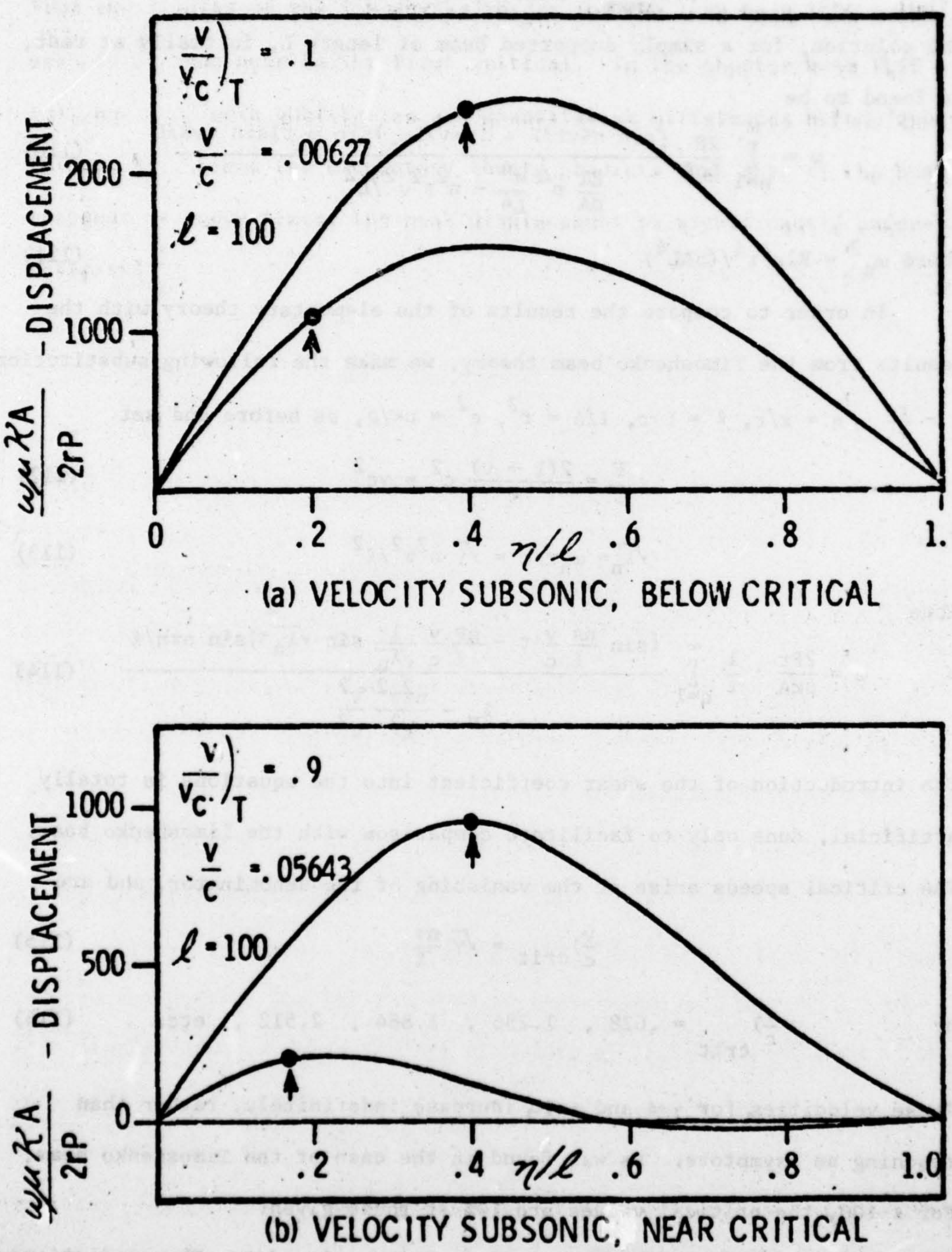
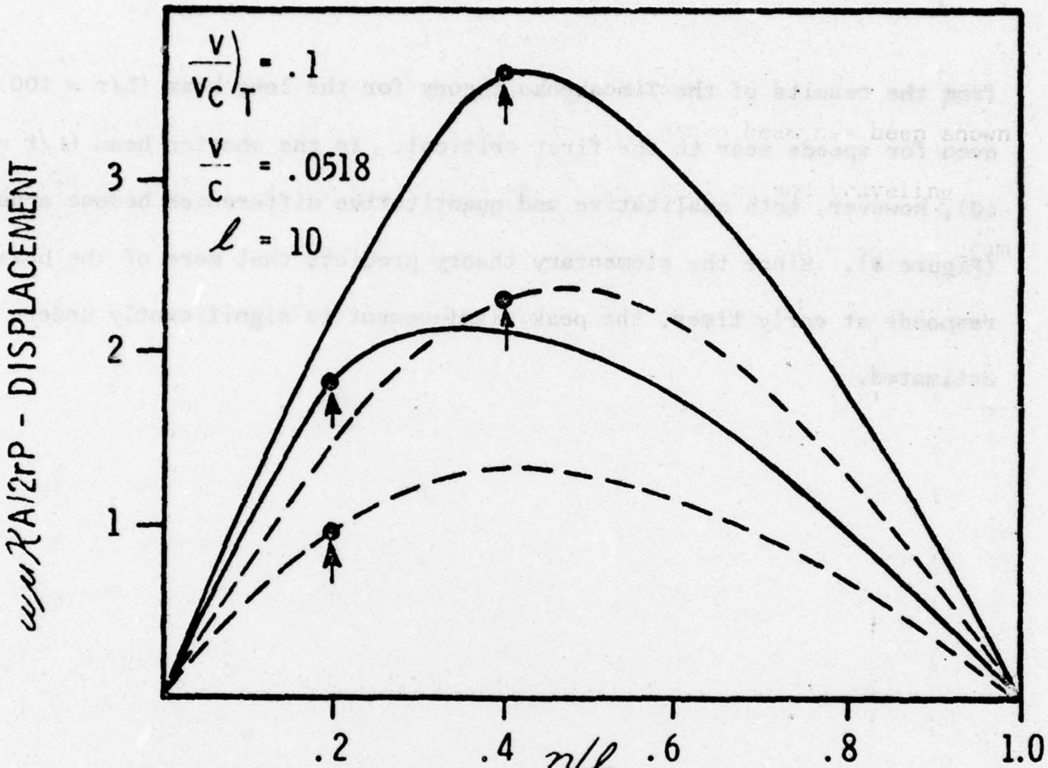
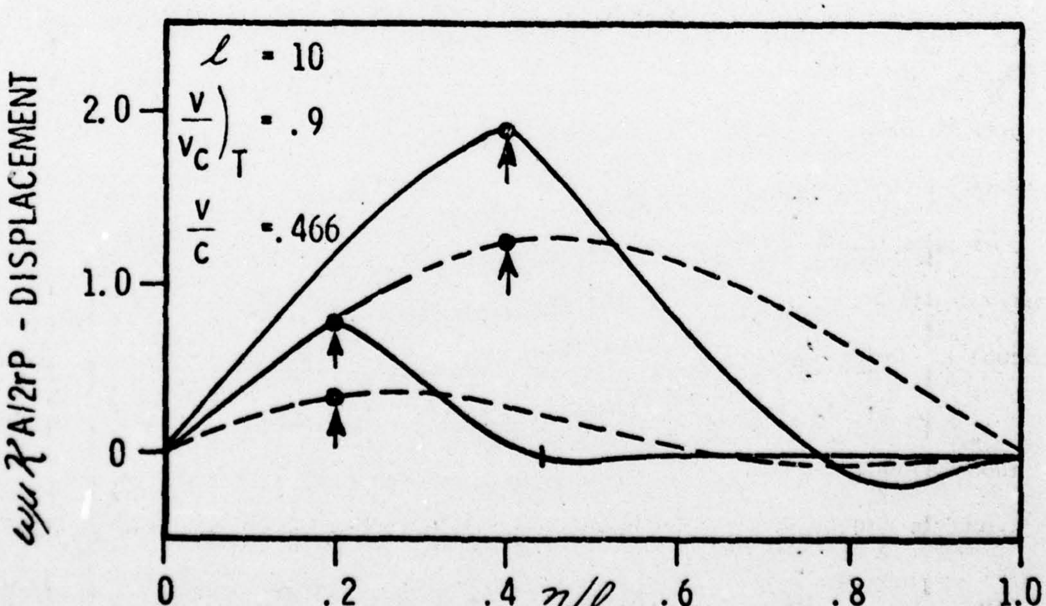


Figure 7 Displacements Due to Traveling Load, Timoshenko or Bernoulli-Euler Beam Theory, ℓ = 100



(a) VELOCITY SUBSONIC, BELOW CRITICAL



(b) VELOCITY SUBSONIC, NEAR CRITICAL

Figure 8 Displacements Due to Traveling Load, Timoshenko cr Bernoulli-Euler Beam Theory, ℓ = 10

from the results of the Timoshenko theory for the long beam (L/r = 100), even for speeds near to the first critical. In the shorter beam (L/r = 10), however, both qualitative and quantitative differences become apparent (Figure 8). Since the elementary theory predicts that more of the beam responds at early times, the peak displacement is significantly underestimated.

IV. Summary

An orthogonality relationship for the Timoshenko beam has been shown to make possible the solution of initial value problems and traveling load problems in a manner analagous to that used for Bernoulli-Euler beam problems. Although only simply supported beams were considered in the examples used, the orthogonality condition and the expansion technique is equally applicable to any of the four possible boundary conditions. Through an example problem (the simply supported beam subjected to a uniform impulse) the expression for the moment in terms of Timoshenko beam modes was found to converge, while the moment expressed in terms of the second derivatives of the displacements of the Bernoulli-Euler beam did not.

A traveling load problem was considered, in which the response of a simply supported Timoshenko beam to a concentrated transverse force, moving at constant speed, was determined and compared to analagous results for the Bernoulli-Euler beam. For the Timoshenko beam, the character of the solution was found to depend on two velocities which are characteristic of the beam. One of these velocities is the effective speed of a shear wave; the other velocity is that velocity of the traveling load which would lead to unbounded motion in a beam of infinite length. For velocities of the moving load which are less than the effective shear speed, no displacement is found ahead of the shear wave. For supersonic loads, i.e., loads moving at velocities greater than the effective shear wave speed, displacements occur ahead of the shear wave front, but the maximum displacement occurs at the front. For very low values of the speed of the moving load, the entire beam is involved in the response, and the response resembles motion in the

first mode, with the peak displacement occurring at, or near the beam center. For larger values of the traveling load, however, the peak displacement is found to occur under the load unless, noted above, the load is supersonic.

For long, thin beams, the predictions of the Bernoulli-Euler theory do not differ significantly from the predictions of the Timoshenko beam theory. For shorter beams however, significant differences are found between the two predictions. The predictions of the elementary theory do not show the previously described tendency for the response to be confirmed to the portion of the beam processed by the shear wave, for such a wave is not predicted by the elementary theory. The predictions of the elementary theory show peak displacements at or near the beam center for velocities of the traveling load where the more complete theory predicts that the peak displacement occurs under the load.

A third significant difference between the results from the two theories is that the critical velocities of the Timoshenko beam were found to have a qualitatively different distribution than the critical velocities of the Bernoulli-Euler beam.

For these reasons, it appears that the Timoshenko beam theory should be used for determining the response to traveling loads for all but the most slender beams. These results also suggest that the influence of shear deformations and rotatory inertia should be considered in determining the response of more complex structural elements, such as rings, plates, and shells to moving loads, for the differences between the predictions of the two theories for beams must be classed as qualitative, rather than as only qualitative. Failure to consider these effects may lead to a failure to model important physical phemomena.

References

1. Timoshenko, S. P. "On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars," Philosophical Magazine, Vol 41, pp 744-746, May 1921.

2. Boley, B. A. and C. C. Chao "Some Solutions of the Timoshemko Beam Equations," Journal of Applied Mechanics, Vol 22, pp 579-586, December, 1955.

3. Plass, H. J. "Some Solutions of the Timoshenko Beam Equations for Short Pulse-Type Loading," Journal of Applied Mechanics, Vol 25, pp 379-385, September, 1958.

4. Anderson, R. A. "Flexural Vibrations in Uniform Beam According to the Timoshenko Theory," Journal of Applied Mechanics, Vol 20, pp 504-510, 1953.

5. Dolph, C. L. "On the Timoshenko Theory of Transverse Beam Vibrations," Quarterly Applied Math, Vol 12, pp 175-187, 1954.

6. Mindlin, R. D. and Deresiewicz, H. "Timoshenko's Shear Coefficient for Flexural Vibrations of Beams," Proc. 2nd National Congress Applied Mechanics, pp 175-178, 1954.

7. Abramsen, H. N., Placc, H. J. and Ripperger, E. A. "Stress Wave Propagation in Rods and Beams," Advances in Applied Mechanics, Vol V, pp 162-175, Academic Press, 1958.

8. Fryba, L. Vibrating Solids and Structures under Moving Loads, Noordhoff International Publishing, Groningen, 1970, pp 361-364.